

INTERSECTION THEOREMS FOR SYSTEMS OF FINITE SETS

By P. ERDŐS, CHAO KO (*Szechuan*), and R. RADO (*Reading*)

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1. Introduction

E. SPERNER (1) has proved that every system of subsets a_ν of a set of finite cardinal m , such that $a_\mu \not\subset a_\nu$ for $\mu \neq \nu$ contains at most $\binom{m}{p}$ elements, where $p = \lfloor \frac{1}{2}m \rfloor$. This note concerns analogues of this result. We shall impose an upper limitation on the cardinals of the a_ν and a lower limitation on the cardinals of the intersection of any two sets a_ν , and we shall deduce upper estimates, in many cases best-possible, for the number of elements of such a system of sets a_ν .

2. Notation

The letters a, b, c, d, x, y, z denote finite sets of non-negative integers, all other lower-case letters denote non-negative integers. If $k \leq l$, then $[k, l]$ denotes the set

$$\{k, k+1, k+2, \dots, l-1\} = \{t: k \leq t < l\}.$$

The obliteration operator $\hat{}$ serves to remove from any system of elements the element above which it is placed. Thus $[k, l] = \{k, k+1, \dots, \hat{l}\}$. The cardinal of a is $|a|$; inclusion (in the wide sense), union, difference, and intersection of sets are denoted by $a \subset b$, $a+b$, $a-b$, ab respectively, and $a-b = a-ab$ for all a, b .

By $S(k, l, m)$ we denote the set of all systems $(a_0, a_1, \dots, a_n)$ such that

$$\begin{aligned} a_\nu &\subset [0, m); |a_\nu| \leq l \quad (\nu < n), \\ a_\mu &\not\subset a_\nu \not\subset a_\mu; |a_\mu a_\nu| \geq k \quad (\mu < \nu < n). \end{aligned}$$

3. Results

THEOREM 1. If $1 \leq l \leq \frac{1}{2}m$; $(a_0, \dots, a_n) \in S(1, l, m)$, then $n \leq \binom{m-1}{l-1}$.

If, in addition, $|a_\nu| < l$ for some ν , then $n < \binom{m-1}{l-1}$.

THEOREM 2. Let $k \leq l \leq m$, $n \geq 2$, $(a_0, \dots, a_n) \in S(k, l, m)$. Suppose that either

$$2l \leq k+m, \quad |a_\nu| = l \quad (\nu < n) \tag{1}$$

$$\text{or}^\dagger \quad 2l \leq 1+m, \quad |a_\nu| \leq l \quad (\nu < n). \tag{2}$$

† The condition $|a_\nu| \leq l$ is in fact implied by $(a_0, \dots, a_n) \in S(k, l, m)$.

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Then (a) either (i)

$$|a_0 \dots a_n| \geq k, \quad n \leq \binom{m-k}{l-k},$$

or (ii) $|a_0 \dots a_n| < k < l < m, \quad n \leq \binom{m-k-1}{l-k-1} \binom{l}{k}^3;$

(b) if $m \geq k + (l-k) \binom{l}{k}^3$, then $n \leq \binom{m-k}{l-k}.$

Remark. Obviously, if $|a_\nu| = l$ for $\nu < n$, then the upper estimates for n in Theorem 1 and in Theorem 2 (a) (i) and (b) are best-possible. For, if $k \leq l \leq m$ and if $a_0, \dots, a_n$ are the distinct sets a such that

$$[0, k) \subset a \subset [0, m), \quad |a| = l,$$

then $(a_0, \dots, a_n) \in S(k, l, m), \quad n = \binom{m-k}{l-k}.$

4. The following lemma is due to Sperner (1). We give the proof since it is extremely short.

LEMMA. *If*

$$n_0 \geq 1, \quad a_\nu \subset [0, m), \quad |a_\nu| = l_0 \quad (\nu < n_0),$$

then there are at least $n_0(m-l_0)(l_0+1)^{-1}$ sets b such that, for some ν ,

$$\nu < n_0, \quad a_\nu \subset b \subset [0, m), \quad |b| = l_0 + 1. \quad (3)$$

Proof. Let n_1 be the number of sets b defined above. Then, by counting in two different ways the number of pairs (ν, b) satisfying (3), we obtain $n_0(m-l_0) \leq n_1(l_0+1)$, which proves the lemma.

5. Proof of Theorem 1

Case 1. Let $|a_\nu| = l$ ($\nu < n$). We have $m \geq 2$. If $m = 2$, then $l = 1$; $n = 1 \leq \binom{m-1}{l-1}$. Now let $m \geq 3$ and use induction over m . Choose, for fixed l, m, n , the a_ν in such a way that the hypothesis holds and, in addition, the number

$$f(a_0, \dots, a_n) = s_0 + \dots + s_n$$

is minimal, where s_ν is the sum of the elements of a_ν . Put $A = \{a_\nu: \nu < n\}$. If $2l = m$, then $[0, m) - a_\nu \notin A$ and, hence

$$n \leq \frac{1}{2} \binom{m}{l} = \binom{m-1}{l-1}.$$

Now let $2l < m$.

Case 1 a. Suppose that whenever

$$m-1 \in a \in A, \quad \lambda \in [0, m) - a,$$

then $a - \{m-1\} + \{\lambda\} \in A$.

We may assume that, for some $n_0 < n$,

$$m-1 \in a_\nu \quad (\nu < n_0), \quad m-1 \notin a_\nu \quad (n_0 \leq \nu < n).$$

Put $b_\nu = a_\nu - \{m-1\} \quad (\nu < n_0)$.

Let $\mu < \nu < n_0$. Then

$$|a_\mu + a_\nu| < 2l < m,$$

and there is $\lambda \in [0, m) - a_\mu - a_\nu$.

Then $b_\mu + \{\lambda\} \in A$, $b_\mu b_\nu = (b_\mu + \{\lambda\})b_\nu = (b_\mu + \{\lambda\})a_\nu \neq \emptyset$

and therefore

$$l-1 \geq 1, \quad (b_0, \dots, b_{n_0}) \in S(1, l-1, m-1).$$

Since $2(l-1) < m-2 < m-1$ we obtain, by the induction hypothesis,

$$n_0 \leq \binom{m-2}{l-2}.$$

$$(a_{n_0}, \dots, a_n) \in S(1, l, m-1), \quad 2l \leq m-1,$$

we have $n - n_0 \leq \binom{m-2}{l-1}$. Thus

$$n = n_0 + (n - n_0) \leq \binom{m-2}{l-2} + \binom{m-2}{l-1} = \binom{m-1}{l-1}.$$

Case 1 b. Suppose that there are $a \in A$, $\lambda \in [0, m) - a$ such that

$$m-1 \in a, \quad a - \{m-1\} + \{\lambda\} \notin A.$$

Then $\lambda < m-1$. We may assume that

$$m-1 \in a_\nu, \quad \lambda \notin a_\nu, \quad b_\nu = a_\nu - \{m-1\} + \{\lambda\} \notin A \quad (\nu < n_0),$$

$$m-1 \in a_\nu, \quad \lambda \notin a_\nu, \quad c_\nu = a_\nu - \{m-1\} + \{\lambda\} \in A \quad (n_0 \leq \nu < n_1),$$

$$m-1 \in a_\nu, \quad \lambda \in a_\nu \quad (n_1 \leq \nu < n_2),$$

$$m-1 \notin a_\nu \quad (n_2 \leq \nu < n).$$

Here $1 \leq n_0 \leq n_1 \leq n_2 \leq n$. Put $b_\nu = a_\nu$ ($n_0 \leq \nu < n$). We now show that

$$(b_0, \dots, b_n) \in S(1, l, m). \quad (4)$$

Let $\mu < \nu < n$. We have to prove that

$$b_\mu \neq b_\nu, \quad b_\mu b_\nu \neq \emptyset. \quad (5)$$

If $\mu < \nu < n_0$ or $n_0 \leq \mu < \nu$, then (5) clearly holds. Now let $\mu < n_0 \leq \nu$. Then $b_\mu \notin A$, $b_\nu = a_\nu \in A$, and hence $b_\mu \neq b_\nu$. If $n_0 \leq \nu < n_1$, then $c_\nu \in A$, and there is $\sigma \in a_\mu c_\nu$. Then $\sigma \neq \lambda$, $\sigma \neq m-1$, and $\sigma \in b_\mu b_\nu$. If $n_1 \leq \nu < n_2$, then $\lambda \in b_\mu b_\nu$. If, finally, $n_2 \leq \nu < n$, then there is

$\rho \in a_\mu a_\nu$. Then

$$\rho \in a_\nu, \quad \rho < m-1, \quad \rho \in b_\mu b_\nu.$$

This proves (5) and therefore (4). However, we have

$$f(b_0, \dots, b_n) - f(a_0, \dots, a_n) = n_0(-[m-1] + \lambda) < 0,$$

which contradicts the minimum property of $(a_0, \dots, a_n)$. This shows that Case 1 *b* cannot occur.

Case 2: $\min(\nu < n) |a_\nu| = l_0 \leq l$.

If $l_0 = l$, then we have Case 1. Now let $l_0 < l$ and use induction over $l - l_0$. We may assume that

$$|a_\nu| = l_0 \quad (\nu < n_0), \quad |a_\nu| > l_0 \quad (n_0 \leq \nu < n), \quad (6)$$

where $1 \leq n_0 \leq n$. Let $b_0, \dots, b_{n_1}$ be the distinct sets b such that (3) holds for some ν . Then

$$(l_0 + 1) - (m - l_0) \leq 2(l - 1) + 1 - m \leq 0, \quad (7)$$

and hence, by the lemma, $n_1 \geq n_0$. Also,

$$(b_0, \dots, b_{n_1}, a_{n_0}, \dots, a_n) \in S(1, l, m).$$

Hence, from our induction hypothesis,

$$n \leq n_1 + (n - n_0) \leq \binom{m-1}{l-1},$$

and Theorem 1 follows.

6. Proof of Theorem 2

Case 1: $k = 0$. Then

$$2l \leq 1 + m, \quad |a_\nu| \leq l \quad (\nu < n), \quad (a_0, \dots, a_n) \in S(0, l, m).$$

Now (a) (ii) is impossible, and (a) (i) is identical with (b), so that all we have to prove is that $n \leq \binom{m}{l}$. Again, we may assume (6), where $l_0 \leq l$, $1 \leq n_0 \leq n$. If $l_0 = l$ then

$$|a_\nu| = l \quad (\nu < n), \quad n \leq \binom{m}{l}.$$

Now let $l_0 < l$ and use induction over $l - l_0$. Let $b_0, \dots, b_{n_1}$ be the distinct sets b such that (3) holds for some ν . Again (7) holds and, by the lemma, $n_1 \geq n_0$. We have

$$(b_0, \dots, b_{n_1}, a_{n_0}, \dots, a_n) \in S(0, l, m)$$

and, by our induction hypothesis,

$$n \leq n_1 + (n - n_0) \leq \binom{m}{l}.$$

This proves the assertion.

Case 2: $k > 0$. We separate this into two cases.

Case 2a. Suppose that (1) holds. Put $|a_0 \dots a_n| = r$. We now show that, if $r \geq k$, then (i) follows. We may assume that

$$a_0 \dots a_n = [m-r, m).$$

Put $a_\nu [0, m-r) = c_\nu \quad (\nu < n)$.

Then $(c_0, \dots, c_n) \in S(0, l-r, m-r)$,

$$2(l-r) - (m-r) = 2l-r-m \leq 2l-k-m \leq 0.$$

Hence, by Case 1,

$$n \leq \binom{m-r}{l-r} = \binom{m-r}{m-l} \leq \binom{m-k}{m-l} = \binom{m-k}{l-k},$$

so that (i) holds. We now suppose that (i) is false, and we deduce (ii).

We have

$$|a_0 \dots a_n| = r < k \leq |a_0 a_1| < |a_0| \leq l,$$

and therefore $2l \leq k+m < l+m$, $k < l < m$.

There is a maximal number $p \geq n$ such that there exist $p-n$ sets $a_n, \dots, a_p$ satisfying $(a_0, \dots, a_p) \in S(k, l, m)$. (8)

Put $A' = \{a_\nu : \nu < p\}$. We assert that

$$(a_0, \dots, a_p) \notin S(k+1, l, m). \quad (9)$$

For otherwise $|a_\mu a_\nu| > k$ ($\mu < \nu < n$). Let $a \in A'$. Then we can choose $a' \subset [0, m)$ such that

$$|a'| = l, \quad |aa'| = l-1.$$

Then, for every $b \in A'$, we have

$$|a'b| \geq |ab| - 1 \geq k$$

and hence, since p is maximal, $a' \in A'$. By repeated application of this result we find that

$$[0, l), [m-l, m) \in A', \quad k < |[0, l)[m-l, m)| = l - (m-l) \leq k,$$

which is the desired contradiction. This proves (9), and hence there are sets $a, b \in A'$ such that $|ab| = k$. Since $|a_0 \dots a_p| \leq |a_0 \dots a_n| < k$, there is $c \in A'$ such that $|abc| < k$. Denote by T the set of all triples (x, y, z) such that $x \subset a, y \subset b, z \subset c, |x| = |y| = |z| = k, |x+y+z| \leq l$. Put $\phi(x, y, z) = \{d : x+y+z \subset d \in A'\}$. Then, by (8),

$$A' = \sum ((x, y, z) \in T) \phi(x, y, z).$$

If $(x, y, z) \in T$ and $s = |x+y+z|$, then $s > k$ since otherwise we obtain the contradiction

$$k > |abc| \geq |xyz| = |x| = k.$$

Hence

$$|\phi(x, y, z)| \leq \binom{m-s}{l-s} = \binom{m-s}{m-l} \leq \binom{m-k-1}{m-l} = \binom{m-k-1}{l-k-1},$$

$$n \leq p = |A'| \leq \binom{m-k-1}{l-k-1} \binom{l}{k}^3,$$

which proves (ii).

Case 2b. Suppose that (2) holds. We may assume (6), where $l_0 \leq l$; $1 \leq n_0 \leq n$. If $l_0 = l$, then Case 2a applies. Now let $l_0 < l$ and use induction over $l - l_0$. Let $b_0, \dots, b_{n_1}$ be the distinct sets b satisfying, for some ν , the relations (3). Then (7) holds and hence, by the lemma, $n_1 \geq n_0$. Also, since $l_0 < l < m$, so that $m - l_0 \geq 2$, we have, by definition of the b_μ ,

$$b_0 \dots b_{n_1} = a_0 \dots a_{n_0}, \quad |b_0 \dots b_{n_1} a_{n_0} \dots a_n| = |a_0 \dots a_n| < k.$$

Since $(b_0, \dots, b_{n_1}, a_{n_0}, \dots, a_n) \in S(k, l, m)$,

it follows from our induction hypothesis that

$$n \leq n_1 + (n - n_0) \leq \binom{m-k-1}{l-k-1} \binom{l}{k}^3.$$

It remains to prove (b) in Case 2. If $k = l$, then (b) is trivial. If $k < l$ and $m \geq k + (l - k) \binom{l}{k}^3$, then

$$\binom{m-k}{l-k} = \binom{m-k-1}{l-k-1} \frac{m-k}{l-k} \geq \binom{m-k-1}{l-k-1} \binom{l}{k}^3,$$

so that (b) follows from (a). This completes the proof of Theorem 2.

7. Concluding remarks

(i) In Theorem 2 (b) the condition

$$m \geq k + (l - k) \binom{l}{k}^3,$$

though certainly not best-possible, cannot be omitted. It is possible for

$$(a_0, \dots, a_n) \in S(k, l, m), \quad k \leq l \leq m$$

to hold and, at the same time, $n > \binom{m-k}{l-k}$. This is shown by the following example due to S. H. Min and kindly communicated to the authors. Let $a_0, \dots, a_n$ be the distinct sets a such that

$$a \subset [0, 8), \quad |a| = 4, \quad |a \cap [0, 4)| = 3.$$

Then

$$n = 16, \quad (a_0, \dots, a_{16}) \in S(2, 4, 8), \quad \binom{m-k}{l-k} = \binom{8}{2} = 15 < n.$$

A more general example is the following. Let $r > 0$ and denote by $a_0, \dots, a_n$ the distinct sets a such that

$$a \subset [0, 4r), \quad |a| = 2r, \quad |a[0, 2r)| > r.$$

Then

$$(a_0, \dots, a_n) \in S(2, 2r, 4r),$$

and we have

$$\begin{aligned} n &= \sum (r < \lambda \leq 2r) \binom{2r}{\lambda} \binom{2r}{2r-\lambda} = \frac{1}{2} \sum (\lambda \leq 2r) \binom{2r}{\lambda} \binom{2r}{2r-\lambda} - \frac{1}{2} \binom{2r}{r}^2 \\ &= \frac{1}{2} \binom{4r}{2r} - \frac{1}{2} \binom{2r}{r}^2. \end{aligned}$$

In this case
$$\binom{m-k}{l-k} = \binom{4r-2}{2r-2},$$

and, for every large r , possibly for every $r > 2$, we have $\binom{m-k}{l-k} < n$.

We put forward the conjecture that, for our special values of k, l, m , this represents a case with maximal n , i.e.

$$\text{If } r > 0, \quad (a_0, \dots, a_n) \in S(2, 2r, 4r),$$

$$\text{then } n \leq \frac{1}{2} \binom{4r}{2r} - \frac{1}{2} \binom{2r}{r}^2.$$

(ii) If in the definition of $S(1, l, m)$ in § 2, the condition $a_\mu \not\subset a_\nu \not\subset a_\mu$ is replaced by $a_\mu \neq a_\nu$ and if no restriction is placed upon $|a_\nu|$, then the problem of estimating n becomes trivial, and we have the result:

Let $m > 0$ and $a_\nu \subset [0, m)$ for $\nu < n$, and $a_\mu \neq a_\nu$, $a_\mu a_\nu \neq \emptyset$ for $\mu < \nu < n$. Then $n \leq 2^{m-1}$, and there are $2^{m-1} - n$ subsets $a_n, \dots, a_{2^m-1}$ of $[0, m)$ such that $a_\mu \neq a_\nu$, $a_\mu a_\nu \neq \emptyset$ for $\mu < \nu < 2^{m-1}$.

To prove this we note that of two sets which are complementary in $[0, m)$ at most one occurs among $a_0, \dots, a_n$, and, if $n < 2^{m-1}$, then there is a pair of complementary sets a, b neither of which occurs among $a_0, \dots, a_n$. It follows that at least one of a, b intersects every a_ν , so that this set can be taken as a_n .

(iii) Let $l \geq 3$, $2l \leq m$, and suppose that

$$a_\nu \subset [0, m), \quad |a_\nu| = l \quad \text{for } \nu < n,$$

and

$$a_\mu \neq a_\nu, \quad a_\mu a_\nu \neq \emptyset \quad \text{for } \mu < \nu < n, \quad \text{and } a_0 \dots a_n = \emptyset.$$

We conjecture that the maximum value of n for which such sets a_ν can be found is n_0 , where

$$n_0 = 3 \binom{m-3}{l-2} + \binom{m-3}{l-3}.$$

A system of n_0 sets with the required properties is obtained by taking all sets a such that

$$a \subset [0, m), \quad |a[0, 3)| \geq 2, \quad |a| = l.$$

(iv) The following problem may be of interest. Let $k \leq m$. Determine the largest number n such that there is a system of n sets a_ν satisfying the conditions

$$a_\mu \neq a_\nu, \quad |a_\mu a_\nu| \geq k \quad (\mu < \nu < n).$$

If $m+k$ is even, then the system consisting of the a such that

$$a \subset [0, m), \quad |a| \geq \frac{1}{2}(m+k)$$

has the required properties. We suspect that this system contains the maximum possible number of sets for fixed m and k such that $m+k$ is even.

(v) If in (ii) the condition $a_\mu a_\nu \neq \emptyset$ ($\mu < \nu < n$) is replaced by $a_\mu a_\nu a_\rho \neq \emptyset$ ($\mu < \nu < \rho < n$), then the structure of the system a_ν is largely determined by the result:

Let $m \geq 2$, $a_\nu \subset [0, m)$ for $\nu < n$, $a_\mu \neq a_\nu$ for $\mu < \nu < n$, and $a_\mu a_\nu a_\rho \neq \emptyset$ for $\mu < \nu < \rho < n$. Then $n \leq 2^{m-1}$, and, if $n = 2^{m-1}$, then $a_0 a_1 \dots a_n \neq \emptyset$, so that the a_ν are all 2^{m-1} sets $a \subset [0, m)$ which contain some fixed number t ($t < m$).

For there is a largest p ($1 \leq p \leq n$) such that

$$a_0 a_1 \dots a_p \in \{a_0, a_1, \dots, a_n\}.$$

If $p = n$, then $a_0 \dots a_n = a_\nu \neq \emptyset$ for some $\nu < n$. If $p < n$, then any two of the $n+1$ distinct sets

$$a_0 a_1 \dots a_p, a_0, a_1, \dots, a_n$$

have a non-empty intersection and hence, by (ii), $n+1 \leq 2^{m-1}$. Different proofs of (v) have been found by L. Pósa, G. Hajós, G. Pollák, and M. Simonovits.

REFERENCE

1. E. Sperner, *Math. Z.* 27 (1928) 544-8.